CE 297: Problems in the Mathematical Theory of Elasticity: Homework III

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In the following, the positive sense is to be assumed when integrating over closed contours unless mentioned otherwise.

1. Expand the function $f(z) = \exp(z)/(1-z)$ in a Taylor series about the point $z_0 = i/2$. What is its radius of convergence?

2. Use the idea of formal multiplication and division of infinite series to write down the first 5 terms of the Maclaurin series of $\tan(z)$.

3. Let $a > 0$. Expand the function $f(z) = z/(a^2 - z^2)$ in a Laurent series in powers of z when (i) $|z| < a$ (ii) $|z| > a$

4. Consider $f(z) = \dfrac{z}{(z-2)(z+i)}$. Expand the function in Laurent series in powers of z when (i) $|z| < 1$ (ii) $1 < |z| < 2$ and (iii) $|z| > 2$. Why is one of these series not like the other two?

5. Evaluate the following integrals

$$\oint_C \frac{1}{\cos 2z}\, dz \qquad \oint_C \tanh z \, dz$$

   where C is the unit circle $|z| = 1$.

6. Analyze the singularities of the function $f(z) = \cot(z)$. Also find the order of any pole singularities.

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7. Show that the function $f(z) = \dfrac{(z-1)^2}{z(z+1)^3}$ has a pole of order 3 at $z = -1$ and find its strength.

8. Classify the singularities of the following functions, their locations, type, etc. Also consider the point at infinity.

   (i) $\dfrac{z^3}{z^2+z+1}$ (ii) $\dfrac{e^{2z}-1}{z^2}$ (iii) $\coth\dfrac{1}{z}$ (iv) $\dfrac{z^{1/3}-1}{z-1}$

9. Can the function

$$\sum_{n=0}^{\infty} z^{2n} \quad |z| < 1$$

   be extended analytically outside the domain of its definition? If so, how?

10. Using cross-cuts and the notion of the residue of a function (coefficient of the $1/(z-z_0)$ term in the Laurent series), prove the following theorem: Let $f(z)$ be holomorphic inside a simple closed contour C and continuous on C, except for a finite number of isolated singular points $z_1, z_2, ...z_N$ which lie inside C. Then,

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{N} a_j$$

   where a_j is the residue of $f(z)$ at $z = z_j$. This result is called the *Residue Theorem.*

11. Evaluate the integral

$$\frac{1}{2\pi i}\oint_C \frac{a^2-z^2}{a^2+z^2}\frac{1}{z}\,dz$$

   where C is any simple closed contour enclosing the poles of the integrand.

12. Evaluate the following real integral using complex variable techniques:

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx$$

   Hint: You will need to close the contour using a semicircle.

13. Evaluate the following real integral using the Residue Theorem:

$$\int_0^{\infty} \frac{1}{(a^2+x^2)^2}\,dx$$

   where $a > 0$.

14. Establish the Schwarz reflection formula for inversion in the circle $|z| = a$ using series.

15. Given the Cauchy integral

$$\Phi(z) = \oint_C \left(t^2 - \frac{1}{t} + \frac{1}{t^2} \right) \frac{dt}{t - z}$$

where C is the unit circle, find $\Phi^+(z)$, $\Phi^-(z)$, and the boundary values $\Phi^+(t)$ and $\Phi^-(t)$.

16. Evaluate the Cauchy Principal Value of the following real integral using the basic definition:

$$\int_{-a}^{a} \frac{s^2}{x - s} dx \qquad -a < x < a \quad a > 0$$

17. Evaluate the Cauchy Principal value of the following integral *without* using the Plemelj formulae:

$$PV \oint_C \frac{dt}{t(t-4)(t-t_0)}$$

where C is the unit circle $|z| = 1$ and $t_0 \in C$. Clearly explain all steps.

18. Verify your result above by computing the left and right boundary values and applying the Plemelj formulae. How will your results change if C is the circle $|z| = 5$?